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NUMERICAL SOLUTION OF THE GOURSAT PROBLEM

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Abstract

The Goursat problem, associated with hyperbolic partial differential equations, arises in several areas of applications. Several finite difference schemes have been proposed to solve the Goursat problem. Amongst these schemes is a scheme which implements harmonic mean averaging of function values. A comparative study which has been carried out concluded that harmonic mean averaging yielded more accurate results than arithmetic mean averaging. However, there seemed to be discrepancies between the conclusions and the displayed results. In this paper, we present the results of a comparative study which we have conducted on three Goursat problems over a range of grid sizes. Our results indicate that arithmetic mean averaging is more accurate than harmonic mean averaging. We also show that arithmetic mean averaging has an advantage when applied to linear Goursat problems.

Key Words

Goursat problem, finite difference schemes, arithmetic mean, harmonic mean

1. Introduction

Many mathematical models in science and engineering necessitate the solution of a partial differential equation. As analytical solutions are often difficult to obtain, numerical methods (such as the finite difference or finite element method) are frequently used.

The Goursat problem, associated with hyperbolic partial differential equations, arises in several areas of physics and engineering. Frisch and Chao [1], Cheung [2], Kaup and Newell [3], An and Hua [4], Hillion [5], McClaughlin et. al. [6], Chen and Li [7], Kaup and Steudel [8] describe in detail areas of applications where a Goursat problem arises. Several finite difference schemes have been proposed to solve the Goursat problem. Amongst these schemes is a scheme which implements harmonic mean averaging of function values. A comparative study which has been carried out [9] concluded that this approach yielded more accurate results compared with the use of the standard method of arithmetic mean averaging.

In this paper we study the accuracy of a finite difference scheme based on harmonic mean averaging and a finite difference scheme based on arithmetic mean averaging when applied to three (one linear, one nonlinear and one with a derivative term) Goursat problems.

2. The Goursat Problem and Finite Difference Schemes

The Goursat problem is of the form [9]:

$$\begin{aligned}&u_{xy} = f(x, y, u, u_x, u_y)\\&u(x,0) = \phi(x), u(0,y) = \psi(y), \phi(0) = \psi(0) \quad (1)\\&0 \le x \le a, 0 \le x \le b\end{aligned}$$

The established finite difference scheme is based on arithmetic mean (AM) averaging of function values and is given by (Wazwaz, 1993):

$$\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{1}{4}\left(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}\right) \quad (2)$$

h denotes the grid size. Henceforth, we shall refer to the finite difference scheme (2) as the AM scheme. For the AM scheme, the function value at grid location $(i+1/2, j+1/2)$ is given by:

$$\frac{1}{4}\left(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}\right) \quad (3)$$

Wazwaz (1993) presented a new scheme for the Goursat problem. This scheme is based on harmonic mean averaging of function values and is given by:

$$\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{4 f_{i+1,j+1} f_{i,j} f_{i,j+1} f_{i+1,j}}{f_{i+1,j+1} f_{i,j}(f_{i+1,j} + f_{i,j+1}) + f_{i+1,j} f_{i,j+1}(f_{i+1,j+1} + f_{i,j})} \quad (4)$$

Henceforth, we shall refer to the finite difference scheme (4) as the HM scheme. The harmonic mean (HM) of any two real numbers a and b is $\frac{2ab}{a+b}$. The function value at location $(i+1/2, j+1/2)$, i.e. the r.h.s of equation (4), is obtained from:

$$\text{HM(HM of } f_{i,j+1} \text{ and } f_{i+1,j} \text{ ;HM of } f_{i+1,j+1} \text{ and } f_{i,j} \text{)} \tag{5}$$

Wazwaz [9] stated that he investigated the application of the AM and HM scheme over a wide range of examples and concluded that the HM scheme appears to give better results (in terms of accuracy). However, results were only presented for the non-linear Goursat problem (with h=0.05):

$$\begin{aligned} &u_{xy} = e^{2u} \\ &u(x,0) = \frac{x}{2} - \ln(1+e^x) \\ &u(0,y) = \frac{y}{2} - \ln(1+e^y) \\ &0 \le x \le 4, 0 \le y \le 4 \end{aligned} \tag{6}$$

The results presented were the relative errors at 16 selected grid points. However, from an examination of the displayed results we observed that the AM scheme was more accurate than the HM scheme <u>for all</u> 16 grid points. The question that arises is whether the conclusion that the HM scheme was more accurate was based on performance at grid points not displayed (for h=0.05, there are 6400 grid points). We aim to compare the accuracy of the AM and HM schemes, for a particular grid size, by computing the number of grid points at which one scheme is more accurate than the other and the average relative error over all grid points. This will be implemented over a range of grid sizes on problem (6) as well as the linear Goursat problem:

$$\begin{aligned} &u_{xy} = u \\ &u(x,0) = e^x \\ &u(0,y) = e^y \\ &0 \le x \le 2, 0 \le y \le 2 \end{aligned} \tag{7}$$

and

$$\begin{aligned} &u_{xy} = -1 + y + u \\ &u\,(x,0) = -1 + e^x \\ &u\,(0,y) = -1 + e^y \\ &0 \le x \le 2.4, 0 \ge y \le 2.4 \end{aligned} \tag{8}$$

In this way we hope to draw firmer conclusions regarding the accuracy of the AM and HM schemes. Analytical solutions for (6), (7) and (8) can be found in Wazwaz [10].

3. Numerical Experiments

Computer programs for problems (6), (7) and (8) were developed. For the non-linear Goursat problem (6) with h = 0.05, we obtained:

Table 1: Relative errors for the AM scheme, h=0.05

y \ x	1.0	2.0	3.0	4.0
1.0	7.2890713e-005	9.7076691e-005	6.4976523e-005	4.0385671e-005
2.0	9.7076691e-005	2.9467327e-004	3.4128209e-004	2.5890833e-004
3.0	6.4976523e-005	3.4128209e-004	8.0424181e-004	9.5132232e-004
4.0	4.0385671e-005	2.5890833e-004	9.5132232e-004	2.1149666e-003

Table 2: Relative errors for the HM scheme, h=0.05

y \ x	1.0	2.0	3.0	4.0
1.0	9.0306034e-005	1.8110830e-004	2.0081184e-004	1.7570651e-004
2.0	1.8110830e-004	5.3529258e-004	7.9519142e-004	7.6182887e-004
3.0	2.0081184e-004	7.9519142e-004	1.9192810e-003	2.5539158e-003
4.0	1.7570651e-004	7.6182887e-004	2.5539158e-003	5.7780549e-003

We also computed that the:

Number of grid points where the AM scheme is superior = 6400

Number of grid points where the HM scheme is superior = 0

Average relative error of the AM scheme = 2.5268253e-004

Average relative error of the HM scheme = 6.2442268e-004

For grid sizes h = 0.025, 0.1, we obtained the following results:

Table 3: Results for Problem (6) with h= 0.025, 0.1

	h = 0.025	h = 0.1
No. of grid points for which AM scheme is superior	25600	1600
No. of grid points for which HM scheme is superior	0	0
Average relative error of the AM scheme	6.2298718e-005	1.0387387e-003
Average relative error of the HM scheme	1.5377398e-004	2.569877e-003

For the linear Goursat problem (7), we obtained the following results:

Table 4: Results for Problem (7) with h=0.025, 0.05, 0.1

	h = 0.025	h = 0.05	h = 0.1
No. of grid points for which the AM scheme is superior	6400	1600	400
No. of grid points for which the HM scheme is superior	0	0	0
Average relative error of the AM scheme	4.3506679e-005	1.7771664e-004	7.4107470e-004
Average relative error of the HM scheme	8.6977635e-005	3.5484552e-004	1.4723821e-003

For the Goursat problem (8), we obtained the following results :

Table 5 : Results for Problem (8) with h = 0.003, 0.006, 0.03

	h = 0.003	h = 0.006	h = 0.03
No. of grid points for which the AM scheme is superior	637647	158706	6190
No. of grid points for which the HM scheme is superior	2353	1294	210
Average relative error of the AM scheme	1.0024714e-3	2.0028735ex-3	9.9341687ex-3
Average relative error of the HM scheme	5.1360557ex-2	5.1946317ex-2	5.6393552ex-2

From the above results (and the results for h values not displayed in this paper) it is clear that the AM scheme is more accurate than the HM scheme.

4. Implementation Aspects

Consider the linear Goursat problem (7) as an example. If the AM scheme is used, we obtain the finite difference scheme:

$$\frac{u_{i+1,j+1}+u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2} = \frac{1}{4}\left(u_{i+1,j+1}+u_{i,j}+u_{i+1,j}+u_{i,j+1}\right) \tag{9}$$

Equation (7) is a linear equation which can easily be solved for the unknown $u_{i+1,j+1}$

the harmonic mean scheme is used we obtain:

$$\frac{u_{i+1,j+1}+u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2} = \frac{4u_{i+1,j+1}u_{i,j}u_{i,j+1}u_{i+1,j}}{u_{i+1,j+1}u_{i,j}(u_{i+1,j}+u_{i,j+1})+u_{i+1,j}u_{i,j+1}(u_{i+1,j+1}+u_{i,j})} \tag{10}$$

This is a non-linear equation in the unknown $u_{i+1,j+1}$ and would require iteration, with its associated computational costs, for its solution. We thus see that the AM scheme has an advantage in that it preserves the linearity of a linear Goursat problem and consequently the straightforward solution procedure.

5. Conclusions

In this paper we have studied the AM and HM finite difference schemes for the solution of the Goursat problem. A previous comparative study concluded that the HM scheme was more accurate. However, the displayed results indicated otherwise. Our investigations, involving the computation of the number of points at which one scheme was more accurate than the other and the comparison of the average relative error for three Goursat problems, have found that the AM scheme is more accurate. We further make the observation that for linear problems the AM scheme has an advantage in that it preserves the linearity of linear Goursat problems.

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References

[1] R.A. Frish and B.R. Chao, On a bounded one-dimensional Poisson-Vlasov System, *SIAM Journal of Applied Mathematics*, 1973.

[2] T.Y. Cheung, Three non-linear ivp's of the hyperbolic type, *SIAM Journal of Numerical Analysis*, 1978.

[3] D.J. Kaup & A.C. Newell, The Goursat and Cauchy problems for the Sine Gordon Equation., *SIAM Journal of Applied Mathematics*, 1978.

[4] Y.L-An, W.C.-Hua, The discontinuous ivp of a reacting gas flow system., *Transactions of the American Mathematical Society*, 1981.

[5] P. Hillion, A note on the derivation of paraxial equation in nonhomogeneous media, *SIAM Journal of Applied Mathematics*, 1992.

[6] J.R. Mclaughlin, P.L. Polyakov, P.E. Sacks, Reconstruction of a spherical symmetric speed of sound, *SIAM Journal of Applied Mathematics*, 1994.

[7] S.Chen and D.Li, Supersonic flow past a symmetrically curved cone, *Indiana University Mathematics Journal*, 2000.

[8] D.J. Kaup and H.Steudel, Virtual Solutions and the asymptotics of second harmonic generation, *Inverse Problems, vol 17*, 2001.

[9] A.M. Wazwaz, On the Numerical solution for the Goursat Problems, *Applied Mathematics and Computation*, 1993.

[10] A.M. Wazwaz, The decomposition method for approximate solution of the Goursat Problem. *Applied Mathematics and Computation*, 1995

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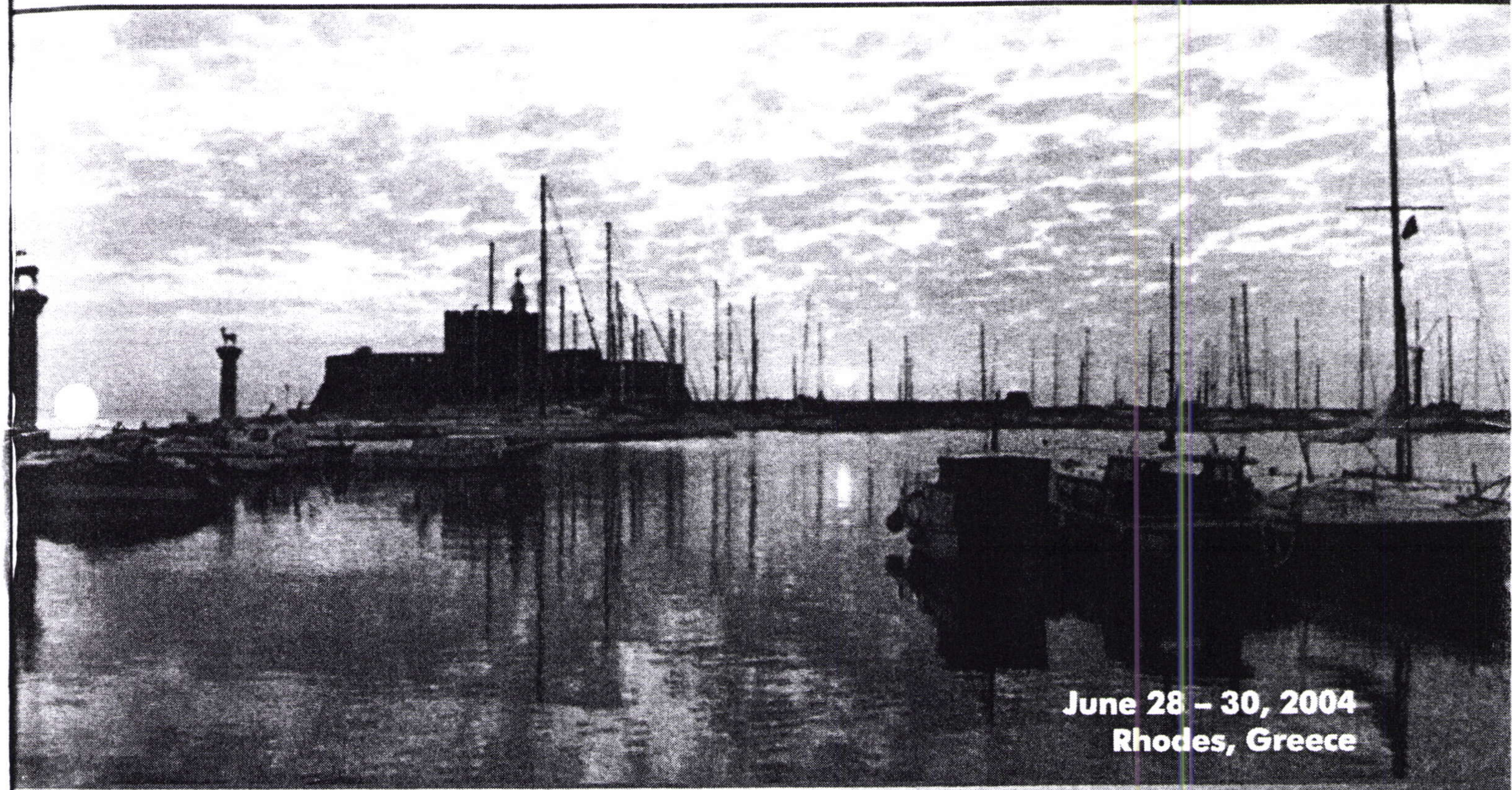

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